SIZES OF INFINITE SETS OF NATURAL NUMBERS
(TAMANHOS DE CONJUNTOS INFINITOS DE NÚMEROS NATURAIS)

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ABSTRACT

In view of the paradox of Euclid's principle, that the part is smaller than the whole, in comparison with Cantor's one-to-one relationship, allowing proper subsets to be placed in a one-to-one relationship with the larger set, I present a refutation of this one-to-one relationship and, preserving, therefore, subject to further consideration, the principle of Euclid, I propose a construction of the natural numbers and an algorithmic definition of the size of finite sets of natural numbers, in addition to definitions of sizes of infinite sets of natural numbers and of sets of non-negative real numbers, bounded or not, which are based on the notion of "asymptotic density" and make use of the notion of limit and the Lebesgue measure regarding subsets of the non-negative real line.

Keywords Cantor · Natural Numbers · Size · Infinite · Paradox

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Conflict of Interest

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1 Introduction

Paolo [5] states, based on Cantor and Hume, that two infinitely denumerable sets that can be placed in one-to-one correspondence have the same "size", that is, the same cardinality. This, however, is contradictory to the principle, dating back to Euclid, that the part is less than the whole. A classic example of this contradiction is the comparison of the set of natural numbers with the subset of even numbers. They can be placed in one-to-one correspondence, for example, through a function that associates each natural number n with the even number $2n$, which must have the same "size" according to the Cantor-Hume principle, but the even numbers, being a proper subset of natural numbers, should have a smaller "size" according to Euclid's principle. In that article ([5]), its author provides us with a historical overview of the issue, which involves deep philosophical discussions of basic concepts, among them the definition of the concept of natural number itself, the concepts of size of infinite sets and the possibility and way to compare them. In passing, the author mentions the possibility of using the notion of "asymptotic density", as used in number theory ([9]), as a possibility to differentiate between infinite sets of natural numbers, despite some limitations. Although different renowned authors have dedicated themselves to the issue of measuring the size of infinite sets of natural numbers, this is an issue that remains controversial. Some prefer to give priority to Euclid's principle while others give primacy to the Cantor-Hume principle. In this article, I address this issue of measuring infinite sets of natural numbers and its

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generalization to sets of non-negative real numbers, making use of the theory of “asymptotic density” and making some philosophical considerations. The entire present theory, as well as the use of the theory of “asymptotic density” to measure infinite sets, I developed independently, but I do not claim any kind of antecedence.

2 Literature Review

According to PAOLO ([5]), questions regarding the very existence and measurability of mathematical infinities have central importance in Western mathematical thought. Paradoxes relating to these issues have been exposed since Antiquity, both in geometric issues and in determining the sizes of infinite collections. The exposure of these paradoxes and the study of the issue regarding measuring the size of infinite collections involved authors such as Galileo, Proclus, Thabit ibn Qurra, Leibniz, Emmanuel Maignan, Bolzano, Cantor, among others ([5]).

Galileo and Leibniz, according to PAOLO ([5, 617]), reject the possibility of a size theory of infinite collections, for different reasons. Maignan “defends the existence of infinite collections and the existence of different sizes among infinities” ([5, 619]). Bolzano and Cantor, in turn, support the possibility of developing a theory of infinite sizes. According to PAOLO([5, 626]), Dedekind approached the definition of infinity through sets that can be placed in one-to-one correspondence with subsets of themselves. As in the case of one-to-one correspondence between natural numbers and even natural numbers. However, according to PAOLO([5, 626]), it was Cantor who used one-to-one correspondences to “analyze the notion of size of infinite sets”.

Thus, for Cantor ([5, 627]), “all infinite sets of natural numbers have the same cardinality, we would say the same ’size’”, since they can be placed in one-to-one correspondence with each other. The problem with this notion is that it conflicts with a highly intuitive principle that goes back to Euclid, in the sense that the part is smaller than the whole PAOLO([5, 639]). Even natural numbers can be placed in one-to-one correspondence with natural numbers and, according to Cantor-Hume, should have the same size. However, since even natural numbers are a proper subset of natural numbers, the collection of the former should be smaller in size than that of the latter, according to Euclid.

There are contemporary attempts to solve this problem, but it is an open question, to the point that Gödel claimed that “Cantor’s theory of infinite set size is inevitable”, since “the number of objects belonging to some class does not change if, by leaving these objects the same, someone changes in some way their properties or mutual relations (for example, their colors or their distribution in space)” PAOLO([5, 637]). Obviously, Gödel would be here referring to the one-to-one relationship between the elements of two infinite sets.

Finally, it is important to mention the use we will make of “asymptotic density”, used in number theory, to measure the size of infinite subsets of natural numbers. This possibility was mentioned in PAOLO ([5, 627]), but I conceived it independently. It should be noted that one can consult [9] and [2] for a definition of this concept.

3 Methodology

In this study, I use the methods of Philosophy in conjunction with those of Mathematics. By methods of Philosophy, I understand, according to [4], the methods of reasoning and analysis that seek to clearly define the concepts used, investigate and expose the foundations of ideas and theories and build a systematic theory that is based on other ideas and systems of thought.

Clearly defining concepts allows you to criticize ideas, compare them, identify their differences and similarities, strengths and weaknesses ([4, 177]). Searching for and exposing the foundations of ideas allows, in turn, a deep critique of the model and the deduction of necessary implications. In this way, these implications can be compared and analyzed ([4, 180]). Aiming to build an organic system must be done in a way that has explanatory and comprehension value. Based on these methods, the work is affiliated with the school of positivism, which makes use of “logic, reason, rigor and inference in the discovery of knowledge” ([4, 189]). It also uses both deduction and induction, therefore it makes both “necessary” and “probable” conclusions ([4, 190]), and seeks both to refute a theory and to construct another.

4 Results and Discussion

4.1 Measuring Infinite Sets of Natural Numbers

4.1.1 Philosophical Considerations

The astonishment surrounding the contradiction between the principles of Cantor-Hume and Euclid is precisely this: both principles are highly intuitive, but, when we accept the first, we are able to construct several one-to-one relationships

between sets and their own subsets and, in this way, we violate the second principle. What principle should remain, then? Furthermore, if we refute the Cantor-Hume principle, which served as the basis for the measurement theory of infinite sets, how can we measure them?

It is necessary to consider that the Cantor-Hume principle applies to any sets and not just to numerical sets. In other words, any two sets have the same "size", the same cardinality, if it is possible to establish a one-to-one relationship between their elements.

As can be seen, this one-to-one relationship, if we consider, firstly, subsets of natural numbers, abstracts, completely disregards, the order of the numbers, that is, the position of each of the numbers in the infinite increasing sequence. However, we argue that natural numbers cannot be defined or thought of without two attributes: order and quantity.

Leaving that aside for now, let's look at how we could define natural numbers. There are, in the literature, several possible definitions of natural numbers ([3]). A recursive, elegant and widely accepted definition is that of John von Neumann: "every number is the set of smaller numbers" ([3, 12]). The first number is the empty set, the next the set whose only element is the empty set, and so on. To provide the definition of natural number to be used here, however, we will need two other primitive notions: nothingness and unity, non-being and being. Nothingness can be intuitively represented by the number 0, that is, no quantity. The unit, whatever it may be, we intuitively represent by the number 1, that is, a unit of quantity. The numbers 0 and 1 are just labels that name these two intuitive notions, these two fundamental quantities, these two primary numbers. It is difficult to imagine a mind that is not based on the basic perception of these two numbers, these two realities: non-existence and existence.

The construction of other natural numbers is an easy result for a mind that is capable of imagining placing a unit next to (alongside) another identical one. This new reality, this union of two units, brings in its essence a new number, distinct from 0 and 1, to which we can attribute the label 2. The other natural numbers can be understood and produced by the same operation of adding, joining, to the last reality, a new unit in sequence. Thus, from the reality underlying the number 2, when we add (join) another unit, we obtain a new reality that we can label 3. Successively in this way, we construct all natural numbers, however large they may be.

Note that such construction and understanding of natural numbers based on non-being and being, nothingness and unity, with the successive additions of new units, has in its essence the notion of order. When we add a unit to a reality, to a number, we build another one that, intuitively, is later and greater than the previous one: it has one more unit. In this construction, order is essential to the notion of number: it is not possible to consider and understand a natural number without relating it to the order, the position it occupies in the sequence. The number 3, for example, can only be considered and understood if we consider that, before it, there are 3 smaller numbers: 0, 1 and 2. And the same can be said about any of the other natural numbers, with the necessary adaptation.

In the same way that, according to this notion of natural number, we cannot consider these numbers without the intrinsic notion of order, we must think that this order is also qualified by a certain "quantity": the unit, the number 1, the numerical label of the being. In other words, any natural number, with the exception of zero, in addition to being greater than the previous number in the increasing sequence, is greater by a specific and constant quantity: the unit, the number 1, the label of being. According to this construction, it is essential that the number 1, this label, can be attributed to a reality and to any other, separate from it, that is essentially identical to it. Only in this way can we construct the number 2 and attribute it to the notion of union of one and another identical reality, when considered together. Only in this way will we be able to construct all natural numbers by the successive addition of new identical realities, new number 1s. In other words, in this construction, natural numbers are inseparable from the notion of order (position) and quantity. Incidentally, we should note that we can define the operation addition $(+)$ of any two natural numbers x and y, simply obtaining, in thought, for each one, its underlying realities, its quantities of units, uniting them, that is, putting them together, in thought, and obtain, in reverse, the label, the number, of this new reality, composed of the union of the realities of the two initial numbers. This last step can be done, starting from 0, from nothingness, counting each of the units in the union as if we were adding them, one by one, to obtain the natural numbers in sequence. We can use, on the other hand, if we have built it, a table that maps any representation of realities to their respective labels and vice versa. The multiplication operation $(\times)$ of natural numbers can be defined based on the addition operation if we can consider that the realities, the units, underlying the numbers can themselves be the realities underlying other numbers (the same number). For example, consider the multiplication of the natural numbers x and y. If we understand that each of the realities (units) of the number x can represent the realities underlying the number y, simply join as many of these new multiple realities (as many numbers y) as there are units of the number x and, from there, inversely map the union of these realities (repetitions of y) to the corresponding label.

Subtracting a smaller natural number from a larger one can easily be done by the inverse action of number construction: instead of successively adding units, we successively remove them (one for each unit of the smaller number) from the union of the units that make up the larger number. From there, simply map the union of realities left over from the

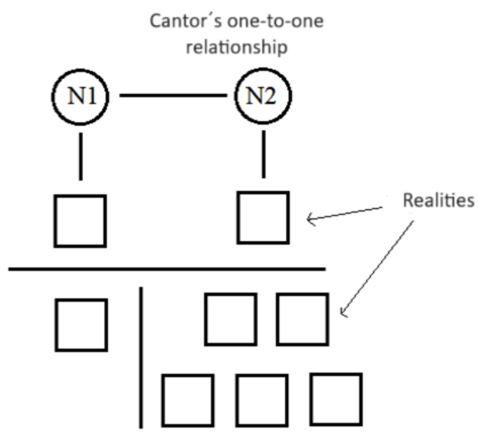

Figure 1: Cantor's One-to-One Relationship.

larger number to the appropriate label. The division of a larger natural number by a smaller one, with the exception of zero, can also be easily defined. For every unit of reality from the smaller number, we take away one unit of reality from the larger number. We note number 1 separately. Repeating this step, if possible, that is, if the number of units of the larger number that remained was equal to or greater (the order relationship, intrinsic in this definition of number, can be easily defined) than the number of units of the smaller number. If so, we add 1 to our annotation. We keep repeating this step. When the number of units of the larger number remaining is less than the number of units of the smaller number, we stop. The division quotient is the number represented in our note. The remainder is obtained by inversely mapping the remaining units of the larger number to the appropriate label.

In this construction, therefore, we cannot separate the notion of a natural number from its position in the ordered sequence of natural numbers. Let's consider an example (See Figure 1, above). When Cantor, in a one-to-one relationship, maps the natural numbers $(0, 1, 2, 3, ...)$ into numbers that are multiples of 5 $(0, 5, 10, 15, ...)$, for example, and abstracts, disregards the labels, as if we mapped $(*, *, *, *, \dots)$ into $(\#, \#, \#, \#, \dots)$, stating that the two sequences have the same "size", we are, in the present construction of natural numbers, making a very serious error, as natural numbers cannot be considered independently of the underlying reality, of their position in the sequence. We can map the 0 of one sequence to the 0 of another. However, we cannot map the number 1 to the number 5, and consider them equivalent, because the underlying realities of both numbers are very diverse, one containing one unit and the other containing $1 + 1 + 1 + 1 + 1$ units. In other words, the number 1 cannot be considered equivalent to the number 5 by a one-to-one relationship in this construction of natural numbers, as this would force us to attribute a label, a number, a unit of reality to two things that are incompatible in terms of number, in quantity terms: one reality unit $(*)$ and $1+1+1+1+1$ reality units $(\#)$. If $(*)$ is equal and equivalent to $(\#)$, both representing a unit of reality, this unit of reality cannot, at the same time, be equivalent to another "one unit" and to $1 + 1 + 1 + 1 + 1$ other "units". When we consider one unit (1) equal to five units $(1 + 1 + 1 + 1 + 1)$, we will have a natural number (as this is what the one-to-one one-to-one relationship is about) that should be equivalent to both 1 and $1 + 1 + 1 + 1 + 1$. But, in this construction, we must map all natural number labels to the same type of reality: the same "pebble" to represent the unit, for example. How, then, is it possible for the number 1 (one unit, $*$ or $\#$), which, in this construction, must be represented by a "pebble", to be equivalent to five $(1+1+1+1+1)$ "pebbles"? If we do this, we will assert that $1+1+1+1+1$ "pebbles", or realities, are equal to 1 other "pebble", or reality. In other words, in this construction, when, with Cantor, we remove the labels and associate a symbol $(*)$ with another symbol $(\#)$ in the one-to-one relationship, we are changing our definition of number: we no longer use "pebbles", but symbols and, we maintain, on the one hand, a symbol $(*)$ essentially equivalent to a "pebble" (1) and, on the other, another symbol, which must also represent a unit (1), replacing 5 $(1 + 1 + 1 + 1 + 1)$ "pebbles". In other words, when we remove the labels and map one (1) symbol onto the other, as a unit, we are lying on one side of the relationship, as a single symbol cannot represent, at the same time, one unit and five units. At least not in questions involving the very definition and construction of natural numbers. We cannot, in Cantor's one-to-one relationship, take the step of considering equivalent, in numerical terms, two natural numbers that represent different quantities of realities. We cannot consider two different natural numbers to be equivalent, in numerical terms. This would require the construction of a meta-notion of natural number that would be equal, on one side of the one-to-one relationship, to one reality, and, on the other side, to five realities, for example. If such a meta-notion existed, it would fail in its application to count the original realities, the "pebbles", since it should make the unit equal, at the same time, to 1 and 5 "pebbles",

which is absurd. This meta-notion, therefore, would be flawed, contradictory, and does not exist. The natural number 1 and the natural number 5, therefore, cannot be made equivalent, in their definition of the notion of unity of reality to which they refer. Therefore, in terms of the definition of number itself, the number 1 cannot be equivalent to the number 5. Different natural numbers cannot be equivalent. We cannot use Cantor's one-to-one relation to, by forcing the equivalence of things (natural numbers) that cannot be equivalent, define a quantity (size, cardinality) that depends on the definition of natural numbers itself. Cantor's cardinality, used in this way, destroys the notion of natural number, of unity of reality, with the purpose of measuring the size of sets of natural numbers.

Reasoning in generic terms (See Figure 1, above), Cantor's one-to-one relationship puts two units of reality into equivalence. Each of these units of reality is, essentially, the intuitive and primitive definition of unity, of being, of the number 1, according to the definition of the present work. We cannot, therefore, affirm that one side of the relationship corresponds to a number, a quantity of realities, distinct from the number, another quantity of realities, on the other side of the relationship. This would be the same as asserting that the unity of reality is, on one side of the relationship, equivalent to one unit of reality and, on the other side of the relationship, to a different number of units of reality. In other words, a unit of reality would be at the same time a quantity different from the unit of reality, which is absurd in terms of the very definition and intuition of number. We therefore disagree with Gödel, as the notions of order and quantity are essential to numbers and cannot be abstracted to support Cantor's one-to-one relationship applied to numbers.

How then can we define the quantity of finite collections of natural numbers? One possible definition, according to the present definition of a natural number, involves a counting algorithm, essentially the same as that used in the definition of number presented above. We start with a counting set, which is initially empty of reality units, and with a counter initially equal to 0. If, in the set to be measured, the number 0, which corresponds to the same void of reality units, is present, we add 1 to our counter. We then add a reality unit to our counting set. If there is a number in the set to be counted, whose number of reality units is the same as the number of realities in our counting set (this number will be 1), we add 1 to our counter. We proceed in this way, and whenever the quantity of reality units of a number in the set to be counted is equal to the quality of reality units in our counting set, we add 1 to our counter. When there are no more numbers in the set to be counted, we end: the counter will contain a number that intuitively represents the number of natural numbers in the finite set that we wanted to count.

4.1.2 Measuring Infinity Using "Asymptotic Density"

As for infinite sets of natural numbers, the standard of measurement, the unit of measurement as we say in Physics, cannot be the unit of reality used to define natural numbers, since, obviously, if we measure an infinite quantity by means of a finite standard of measurement, we will have an infinite quantity, which doesn't say much. The solution, therefore, is to use a unit of measurement that is infinite. As the set of all natural numbers is infinite and is the largest of all sets of natural numbers, if we use it as a basis of measurement, we can compare the size of infinite sequences of natural numbers, to be defined below, with the size of the set of all natural numbers and, in this way, obtain a fraction, to be defined (the concept of fraction can be defined from ordered pairs of natural numbers), as a measure of the size of the sequence of natural numbers in question.

So that we can capture the notion of infinity, we use the notion of limit (to be defined) of a sequence of fractions; and, so that we can define the quantity of an infinite sequence of natural numbers, we will use the following definition: $q(n)$ is the quantity of natural numbers of the infinite sequence in question, smaller than or equal to a certain finite natural number n. In the same sense, $Q(n)$ is defined as the quantity of any (natural numbers, less than or equal to a certain finite natural number n. We define the size of an infinite sequence of natural numbers as: $\lim_{n\to\infty}(q(n)/Q(n))$.

This definition corresponds, in essence, to the definition of "asymptotic density" ([9]). It can be seen that, according to this definition, the size of the set of even natural numbers will be $1/2\,(half\,\infty)$ the size of the set of natural numbers $(1\,\infty)$, preserving Euclid's principle and the intuition regarding the sizes of these sets. The same can be verified with respect to several other infinite subsets of natural numbers.

Without intending to elaborate the reasoning around the potential of this definition, it seems intuitive that, since the standard of measurement of natural sets, finite and infinite, is different according to the present work, if we have two infinite sets of natural numbers A and B, such that A is a proper subset of B, and the difference between them is a finite set (of size $n > 0$), we must consider that the measure of A is smaller than that of B, according to the principle of Euclid, and not both with measure equal to $(1\,\infty)$, but with the measure of B being n units greater than that of A.

In the same sense, regarding the set of non-negative real numbers, if we define as a measurement standard or unit of measurement, the length of the interval $[0, 1)$, $l = 1 - 0 = 1$, we can intuitively define the size of a limited set ("E") of real numbers like: $\mu(E)/1$. Where $\mu(E)$ is the Lebesgue measure of this set ([6]). Regarding an unlimited set of non-negative real numbers (F), perhaps we can define its size as: $\lim_{n\to\infty}(\mu(F, r)/r)$.

Where $\mu(F, r)$ is the Lebesgue measure of the subset of F less than or equal to r, an increasing real number. In the latter case, the infinite measurement standard would be the set of non-negative real numbers itself.

5 Final Considerations

The study carried out here attacks the core of Cantor's one-to-one relationship, preserving Euclid's principle, that the part is smaller than the whole, which always seemed more intuitive to me. Furthermore, I propose an algorithmic construction of natural numbers and an algorithmic definition of the size of finite natural sets, which does not depend on Cantor's one-to-one relationship. I also intuitively propose a definition of the size of infinite sets of natural numbers and definitions of the size of subsets, limited or not, of non-negative real numbers, which also do not depend on Cantor's one-to-one relationship. The definitions, in all cases, preserve the intuitions of Euclid's principle and set size, but have not been verified except for a few very simple examples of sets. Therefore, the present theory should be refined, tested for its functionality and usefulness, as well as developed in all its implications.

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